RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2602

Pure Mathematics 2

Monday **23 MAY 2005** Morning 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer **all** questions.
- You are permitted to use only a scientific calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 4 printed pages.

1 **(a)** Differentiate $\dfrac{x}{1 + \ln x}$. [3]

(b) Differentiate $\sqrt{1+x^3}$. [3]

(c) **(i)** Given that $y = 1 + x^{\frac{1}{3}}$, find $\dfrac{dy}{dx}$.

(ii) Using your result, find $\dfrac{dx}{dy}$ in terms of x.

(iii) Find x in terms of y. By differentiating your result, verify your answer to part **(ii)**. [6]

(d) Show that $\displaystyle\int_0^{a^2} (1 + 3\sqrt{x})\,dx = a^2(1+2a)$. [3]

[Total 15]

2 **(i)** An arithmetic progression has first term -8. The 20th term is three times the 10th term. Find the common difference. [3]

(ii) Another arithmetic progression has common difference 2. The sum of the first 20 terms is three times the sum of the first 10 terms. Find the first term. [4]

(iii) A geometric progression is such that its 20th term is three times its 10th term. The first term is not zero, and the common ratio is positive. Find the common ratio, giving your answer to 3 significant figures. [3]

(iv) Another geometric progression has non-zero first term and common ratio r, where $r > 0$ and $r \neq 1$. The sum of the first 20 terms is three times the sum of the first 10 terms. Show that

$$u^2 - 3u + 2 = 0,$$

where $u = r^{10}$. Hence find the value of r. [5]

[Total 15]

3 Fig. 3 shows the graph of $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x\mathrm{e}^{-\frac{1}{2}x^2}.$$

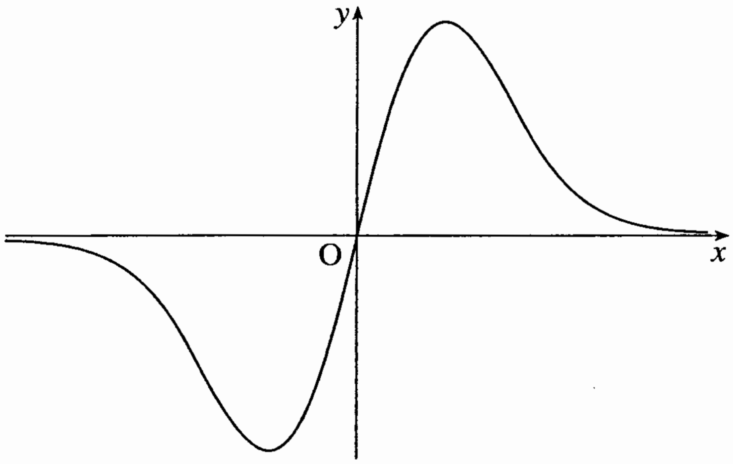

Fig. 3

(i) The curve has rotational symmetry about the origin. Name the type of function that has this property, and show algebraically that $\mathrm{f}(x)$ is such a function. [3]

(ii) Differentiate $\mathrm{e}^{-\frac{1}{2}x^2}$. Hence show that $\mathrm{f}'(x) = (1 - x^2)\mathrm{e}^{-\frac{1}{2}x^2}$. [4]

(iii) Find the exact coordinates of the stationary points of the curve. [4]

(iv) The area of the finite region enclosed by the x-axis, the curve and the line $x = 1$ is A. Using the substitution $u = \frac{1}{2}x^2$, show that

$$A = \int_0^{\frac{1}{2}} \mathrm{e}^{-u}\,\mathrm{d}u.$$

Evaluate A exactly. [5]

[Total 16]

4 **(i)** The variables x and y are related by the equation

$$y = a \times b^x,$$

where a and b are constants.

Show that plotting $\ln y$ against x produces a straight line graph. State the gradient of the line and the intercept with the $\ln y$ axis. [3]

Two relationships of this form are illustrated by the lines l and l' in Fig. 4.

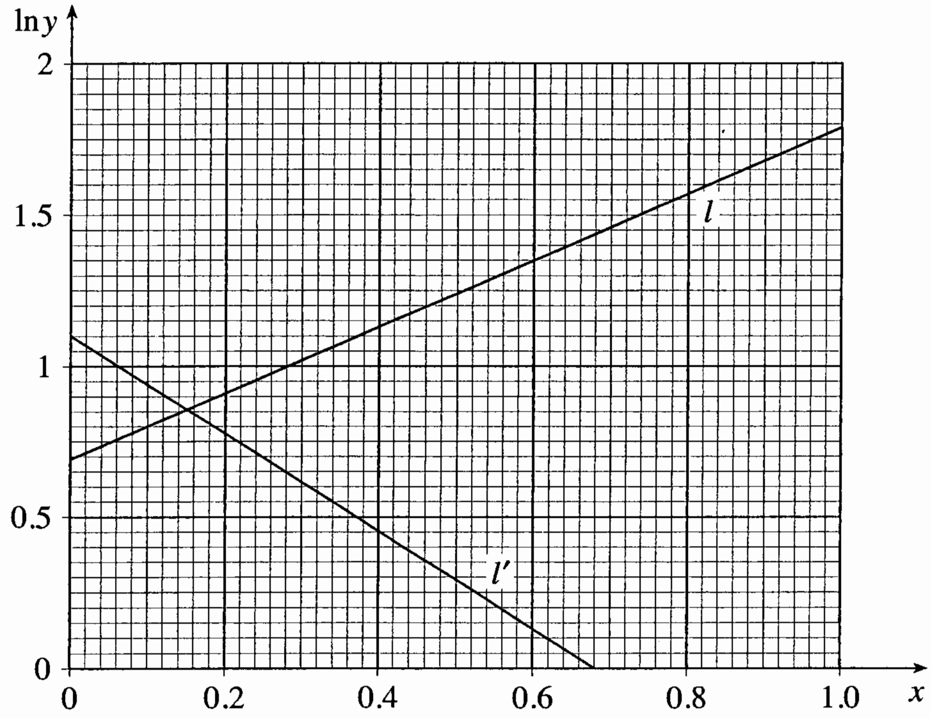

Fig. 4

Line l is the result of plotting $\ln y$ against x when $y = 2 \times 3^x$.

(ii) Verify that the line l has the correct intercept with the $\ln y$ axis. [2]

Line l' is the result of plotting $\ln y$ against x when $y = 3 \times c^{-x}$. The constant c is a whole number.

(iii) Use the graph to find the value of the whole number c. [4]

(iv) Use the graph to estimate the value of x which satisfies the equation

$$2 \times 3^x = 3 \times c^{-x}.$$ [1]

(v) Re-arrange this equation to show that

$$x = \frac{\ln 3 - \ln 2}{\ln c + \ln 3}.$$

By substituting the value of c found in part **(iii)**, find a more accurate solution to the equation in part **(iv)**. [4]

[Total 14]

Mark Scheme 2602
June 2005

1 (a) $y = \dfrac{x}{1+\ln x}$ $\dfrac{dy}{dx} = \dfrac{(1+\ln x).1 - x.\frac{1}{x}}{(1+\ln x)^2}$ $= \dfrac{\ln x}{(1+\ln x)^2}$.	M1 M1 A1 [3]	d/dx (ln x) = 1/x soi correct expression using product or quotient rule simplified numerator
(b) $y = (1+x^3)^{\frac{1}{2}}$ let $u = 1 + x^3$, $y = u^{1/2}$ $\Rightarrow \dfrac{dy}{dx} = \dfrac{dy}{du}.\dfrac{du}{dx}$ $= ½\, u^{-1/2}.\ 3x^2$ $= \dfrac{3}{2}x^2(1+x^3)^{-\frac{1}{2}}$.	M1 M1 A1cao [3]	chain rule ½ $u^{-1/2}$ soi or equivalent
(c) (i) $y = 1 + x^{1/3}$ $\Rightarrow$ dy/dx = 1/3 $x^{-2/3}$	B1	isw
(ii) dx/dy = $\dfrac{1}{dy/dx}$ $= 3x^{2/3}$	M1 A1ft	1/ their c(i) correct expression in terms of x **If (iii) not done** scB1 for $\Rightarrow$ dx/dy = $3(y-1)^2$ and scB2 for dx/dy = $3x^{2/3}$
(iii) $x^{1/3} = y - 1$ $\Rightarrow x = (y-1)^3$ $\Rightarrow$ dx/dy = $3(y-1)^2$ $= 3\,(x^{1/3})^2 = 3x^{2/3}$ as before	M1 A1 E1 [6]	raising to power 3 correctly
(d) $\int_0^{a^2}(1+3\sqrt{x})dx$ $= \left[x + 3\dfrac{x^{3/2}}{3/2}\right]_0^{a^2}$ $= \left[x + 2x^{3/2}\right]_0^{a^2}$ $= a^2 + 2a^3\ (-0)$ $= a^2(1+2a)$*	M1 M1 E1 [3]	correctly integrated limits substituted correctly into some attempt at integration www intermediate step (unfactorised form) must be seen for this

2 (i) $a = -8$ $-8 + 19d = 3(-8 + 9d)$ $= -24 + 27d$ $\Rightarrow 16 = 8d$ $\Rightarrow d = 2$	M1 M1 A1 [3]	for either -8 +19d or -8 + 9d their u_{20} = 3 their u_{10} cao (B3 ans without wrong working, verified)
(ii) $\frac{20}{2}(2a+19\times 2) = 3\times\frac{10}{2}(2a+9\times 2)$ $\Rightarrow 10(2a + 38) = 15(2a + 18)$ $\Rightarrow 20a + 380 = 30a + 270$ $\Rightarrow 110 = 10a$ $\Rightarrow a = 11$	M1 M1 A1 A1 [4]	correct expression for either sum their S_{20} = 3 their S_{10} correct equation cao
(iii) $a \times r^{19} = 3 \times a \times r^9$ $\Rightarrow r^{10} = 3$ $\Rightarrow r = 3^{1/10} = 1.12$ (3 s.f.)	M1 A1 B1 [3]	for $a \times r^{19}$ or $a \times r^9$ soi or correct eqn using $a \times r^{19}$ and $a \times r^9$ eg log cao (B3 without working ans 1.12)
(iv) $\frac{a(r^{20}-1)}{r-1} = 3\frac{a(r^{10}-1)}{r-1}$ $\Rightarrow r^{20} - 1 = 3(r^{10} - 1)$ $\Rightarrow r^{20} - 3r^{10} + 2 = 0,\ u = r^{10}$ $\Rightarrow u^2 - 3u + 2 = 0$ * $\Rightarrow (u - 2)(u - 1) = 0$ $\Rightarrow u = 2$ or $u = 1$ $\Rightarrow r^{10} = 2$ or $r^{10} = 1$ $\Rightarrow r = 2^{1/10}$ (= 1.07) ($r \neq 1$)	M1 M1 E1 B1 B1 [5]	for S_{20} or S_{10} their S_{20} = 3 their S_{10} u=2 or $r^{10} = 2$ cao

3 (i) Odd function $f(-x) = -xe^{-(-x)^2/2} = -xe^{-x^2/2} = (-f(x))$	B1 M1 E1 [3]	$f(-x)$ $= -f(x)$ brackets or comment needed to convince re signs
(ii) $y = e^{-x^2/2}$ let $u = -x^2/2$, $du/dx = -2x/2 = -x$ $y = e^u$, $dy/du = e^u$ $\Rightarrow \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = -x\, e^u = -xe^{-x^2/2}$ $f'(x) = x.(-xe^{-x^2/2}) + 1.\ e^{-x^2/2}$ $= (1-x^2)\ e^{-x^2/2}$ *	M1 A1 M1 E1 [4]	chain rule s.o.i. product rule (ft) s.o.i. www
(iii) $(1-x^2)\ e^{-x^2/2} = 0$ $\Rightarrow 1 - x^2 = 0$ $\Rightarrow x^2 = 1$ $\Rightarrow x = 1$ or -1 When $x = 1$, $y = e^{-1/2}$ When $x = -1$, $y = -e^{-1/2}$	M1 A1 A1 A1 [4]	$1 - x^2 = 0$ or first line $=0$ $x = 1$ $y = e^{-1/2}$ $(-1, -e^{-1/2})$ SC A1 for both y-coords decimal only, 0.61 or better
(iv) $A = \int_0^1 xe^{-\frac{1}{2}x^2}\,dx$ let $u = ½\,x^2$, $du/dx = x$ $\Rightarrow du = x\,dx$ When $x = 0$, $u = 0$ When $x = 1$, $u = ½$ $\Rightarrow A = \int_0^{1/2} e^{-u}\,du$ * $= \left[-e^{-u}\right]_0^{1/2}$ $= -e^{-1/2} + 1 = 1 - e^{-1/2}$.	M1 M1 E1 M1 A1 [5]	correct integral (condone missing limits and *dx)* dealing with dx change of limits shown (convincing recovery needed from no *dx*) $\left[-e^{-u}\right]$ or equivalent (no decimals)

4 (i) $y = a \times b^x$ $\Rightarrow$ $\ln y = \ln a + x \ln b$ c.f. $y \;=\; c \;+ x\; m$ gradient = $\ln b$, $\ln y$ – intercept = $\ln a$	M1 B1 B1 [3]	condone log allow m= if M1 scored allow c = if M1 scored otherwise need gradient and intercept
(ii) $b = e^{0.7} = 2.01 \approx 2$ Intercept = 0.7	B1 B1 [2]	ln 2 = 0.69 "which fits" or indication of checking with graph
(iii) Gradient = $-\dfrac{1.1}{0.68} = -1.62 = -\ln c$ $\Rightarrow$ $c = e^{1.6} = 5$ (to nearest whole no)	M1 M1 A1 A1cao [4] M1 M1 A1 A1 M1 M1 A1 A1	gradient = +/- ln c soi using graph values to obtain gradient +/–1.6(2) **OR** lny = ln3 – x ln c substituting a point from graph lnc = numerical expression cao **OR** manipulating equation without lns substituting a point from graph and calculating value of y from value of lny c = numerical expression cao
(iv) 0.15	B1 [1]	
(v) $2 \times 3^x = 3 \times c^{-x}$ $\Rightarrow$ $\ln 2 + x \ln 3 = \ln 3 - x \ln c$ $\Rightarrow$ $x \ln 3 + x \ln c = \ln 3 - \ln 2$ $\Rightarrow$ $x(\ln 3 + \ln c) = \ln 3 - \ln 2$ $\Rightarrow$ $x = \dfrac{\ln 3 - \ln 2}{\ln c + \ln 3}$ * $= \dfrac{\ln 3 - \ln 2}{\ln 5 + \ln 3}$ = 0.1497	M1 M1 E1 B1ft [4]	taking lns at any stage collecting x's at any stage factorisation seen ft integer value of c accept 0.15

2602 - Pure Mathematics 2

General Comments

This paper gave opportunities for candidates of all standards and attracted the full range of marks. Although there were fewer candidates scoring the very highest marks, 55 or more out of the maximum 60, than in previous years, and a significant number scoring less than 10 marks, it appeared that the majority of candidates were familiar with the methods and techniques required by the questions.

There were some very good performances with well presented scripts, and little evidence of candidates being short of time.

There were several points where success depended on careful reading of the question to identify what was required. The request for exact answers in Question 3 was often overlooked or misunderstood and in all questions marks were frequently lost through slips in algebra.

Comments on Individual Questions

1) Question 1 gave opportunities for most candidates, who tended to score well here. The quotient rule and the chain rule were well applied.

In part (a) some weaker candidates misidentified *u* and *v*, and errors in simplifying the answer were quite common. Only a small minority used the product rule.

In part (b) it was good to see that only a very few candidates used the mistaken form $f'\,g'(x)$, in this case $\frac{1}{2}(3x^2)^{-\frac{1}{2}}$, instead of $f'\,g(x).g'(x)$.

Part (c) saw slips and elementary errors of algebra and the final verification in (iii) was often not achieved.

A surprising number went from $y-1=x^{\frac{1}{3}}$

to $y^3-1=x$

and $x=(y-1)^{\frac{1}{3}}$ was often seen.

Another error was to put $(x^{-\frac{2}{3}})^{-1}$ equal to $x^{\frac{3}{2}}$

A few candidates began (iii) by swapping *x* and y, as if looking for an inverse function, which led to confusion.

Part (d) was usually done well but many candidates misread the integrand as $1+x^{\frac{1}{3}}$ The Mark Scheme was generous towards this. Others used unhelpful substitutions.

2) Question 2 was the least well answered question. Whilst there were many excellent solutions, weaker candidates made algebraic slips or had trouble remembering the formulae.

A common error for the sum formula in part (ii) was $\frac{1}{2}n\left[a+(n-1)d\right]$.

Setting up an equation often defeated them, and many ended up with, for example

$$3\times 10^{th} \text{ term} = 10^{th} \text{ term}$$

In part (i), errors such as $3(-8 + 9d) = -24 + 9d$ were seen.

In part (ii) $3(ar^9)=3a\times 3r^9$ was common.

Many of those who established $3ar^9=ar^{19}$ could not solve this equation.

In part (iv) able candidates quickly reduced the initial equation to $r^{20}-1=3(r^{10}-1)$ in one step, to their advantage.

A small number of the stronger candidates misread the question and answered

$$3\times 20^{th} \text{ term} = 10^{th} \text{ term}$$

and 3 x sum of first 20 terms = sum of first 10 terms throughout.

3) Question 3 was generally well answered, although the candidates' algebra was often not up to scoring full marks in part (i). Most candidates knew that the function was odd, but some attempted to verify this by substituting particular values rather than (-*x*).

The differentiation in part (ii) was done well by many.

Some candidates confused f(x) with $\mathrm{e}^{-\frac{1}{2}x^2}$ and so thought that the result of

$\mathrm{e}^{-\frac{1}{2}x^2}$ differentiating should be $(1-x^2)\mathrm{e}^{-\frac{1}{2}x^2}$

In part $\sqrt{1}$ (iii) appeared in many solutions unresolved, and many gave decimal approximations for the y- coordinates, rather than the exact forms.

In part (iv), the technique of integration by substitution was widely understood, but many candidates, having obtained the correct expression, were unable to integrate correctly.

$\int \mathrm{e}^{-u}\,du$ was often given as $\dfrac{\mathrm{e}^{-u}}{-u}$ or $\dfrac{\mathrm{e}^{-u+1}}{-u+1}$ or $\dfrac{\mathrm{e}^{-2u}}{2}$

4) Perhaps it was because this was the last question, and required some careful thinking, that marks were lost by many candidates. Nevertheless, some weaker candidates, who were familiar with the topic, were able to make up ground here and the question was well answered by those confident about the logarithmic notation and laws. There were frequent slips and errors from others.

In part (i) $\ln y = \ln a \times x \ln b$ was common and the correct form, written as $\ln y = x \ln b + \ln c$ led many to offer x as the gradient.

Part (ii) was done well.

In part (iii) mistakes arose from misreading scales, reading from the wrong line, problems with signs and generally whether to use x, y, $\ln x$, $\ln y$, c or $\ln c$.

In part (iv) some wasted time trying to solve rather than estimating from the graph; part (v) was generally done well.